Positivity Phenomena for Lattice Paths at $q = -1$

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Lattice Paths and Symmetric Functions

Delta conjectures/theorems relate operators $\Delta_{e_{n-k}}$ and $\Delta'_{e_{n-k}}$ to the combinatorics of *decorated lattice paths*.

Conjecture (special case of Haglund–Remmel–Wilson [3]): For all n, k,

$$\langle \Delta'_{e_{n-k-1}} e_n, h_1^n \rangle = \sum_{P \in stLD(n)^{\bullet k}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)},$$

where $\mathsf{stLD}(n)^{\bullet k}$ is the set of *standardly-labeled, valley-decorated Dyck paths of semilength n with k decorations.*

Conjecture (special case of D'Adderio–Iraci–Vanden Wyngaerd [2]):

For all n, k,

$$\frac{[n-k]_q}{[n]_q} \langle \Delta_{e_{n-k}} \omega p_n, h_1^n \rangle = \sum_{P \in \mathsf{stLSQ}(n)^{\bullet k}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)},$$

where $\mathsf{stLSQ}(n)^{\bullet k}$ is the set of *standardly-labeled, valley-decorated square paths of semilength n with k decorations.*

Both conjectures reduce to known results when $k = 0$ ($\Delta_{e_n} = \nabla$).

$\mathsf{area}(P) = \#$ of whole squares between path and lowest diagonal.

$\mathsf{dinv}(P) = \#$ of "diagonal inversions".

Specializations

We're interested in behavior at $q = -1$.

$$D_{n,k} := \sum_{P \in stLD(n)^{\bullet k}} (-1)^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)}$$

$$S_{n,k} := \sum_{P \in stLSQ(n)^{\bullet k}} (-1)^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)}$$

Surprisingly, these are *positive* polynomials!

Theorem (Corteel–Josuat-Vergès–Vanden Wyngaerd [1]):

$$\sum_{k=0}^{n-1} D_{n,k} z^k = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathsf{inv}_3(\sigma)} z^{\mathsf{monot}(\sigma)},$$

where inv_3 and monot are certain permutation statistics.

In particular, at $z = 0$, we have

$$\langle \nabla e_n, h_1^n \rangle|_{q=-1} = D_{n,0} = t^{\lfloor n^2/4 \rfloor} E_n(t),$$

where $E_n(t)$ is a t-analog of the Euler numbers counting alternating permutations according to $31-2$ patterns.

Decorated Square Paths

(a) An element of $\mathsf{LSQ}(7)^{\bullet 3}$. (b) An element of $\mathsf{stLSQ}(7)^{\bullet 3}$.

Figure 1. Valley-decorated, labeled square paths.

Square Paths at $q = -1$

Square path enumerator *also* behaves nicely at $q = -1$:

Theorem:

$$\sum_{k=0}^{n-1} S_{n,k} z^k = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathsf{revmaj}(\sigma)} z^{\mathsf{parity-dec}(\sigma)},$$

where $\mathsf{revmaj}(\sigma)$ is the major index of the reverse of σ and $\mathsf{parity-dec}(\sigma)$ is a more complicated statistic.

When $z = 0$, we have

$$\langle \nabla \omega p_n, h_1^n \rangle|_{q=-1} = S_{n,0} = \begin{cases} [n]_t t^{\lfloor (n-1)^2/4 \rfloor} E_{n-1}(t), & n \text{ odd}, \\ 0, & n \text{ even}, \end{cases}$$

so setting $t = 1$ gives

$$\sum_{P \in \mathsf{stLSQ}(n)} (-1)^{\mathsf{dinv}(P)} = \begin{cases} nE_{n-1}, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

Square Paths and Dyck Paths

The square path results are closely related to the Dyck path results:

Theorem:

$$\sum_{k=0}^{n-1} S_{n,k} = \sum_{k=0}^{n-1} D_{n,k} = [n]_t!.$$

Moreover,

$$S_{n,k} = \begin{cases} [n]_t (D_{n-1,k} + D_{n,k-1}), & n-k \text{ odd}, \\ 0, & n-k \text{ even}. \end{cases}$$

Relationship is proved by giving an explicit bijection.

Proof Techniques & Cyclic Actions

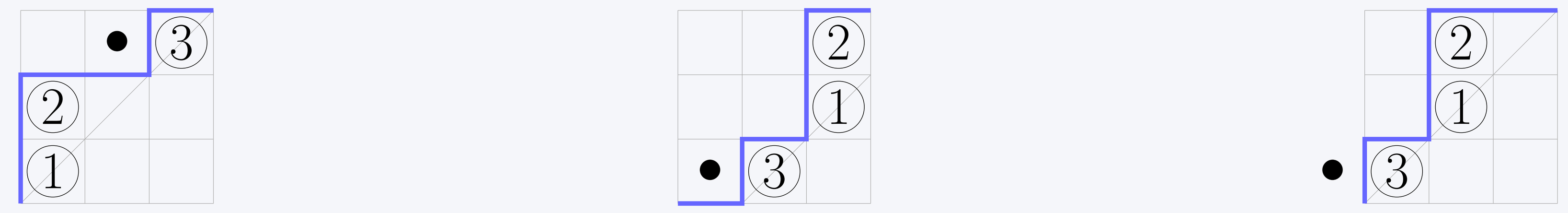
Proofs rely on *schedule numbers*: n-tuple of nonneg. integers assigned to each lattice path.

When $q = -1$, contribution of P to $S_{n,k}$ (or $D_{n,k}$) vanishes *unless* $\mathsf{sched}(P) = 1^n$.

For $S_{n,k}$, need to combine this with a cyclic action on square paths.

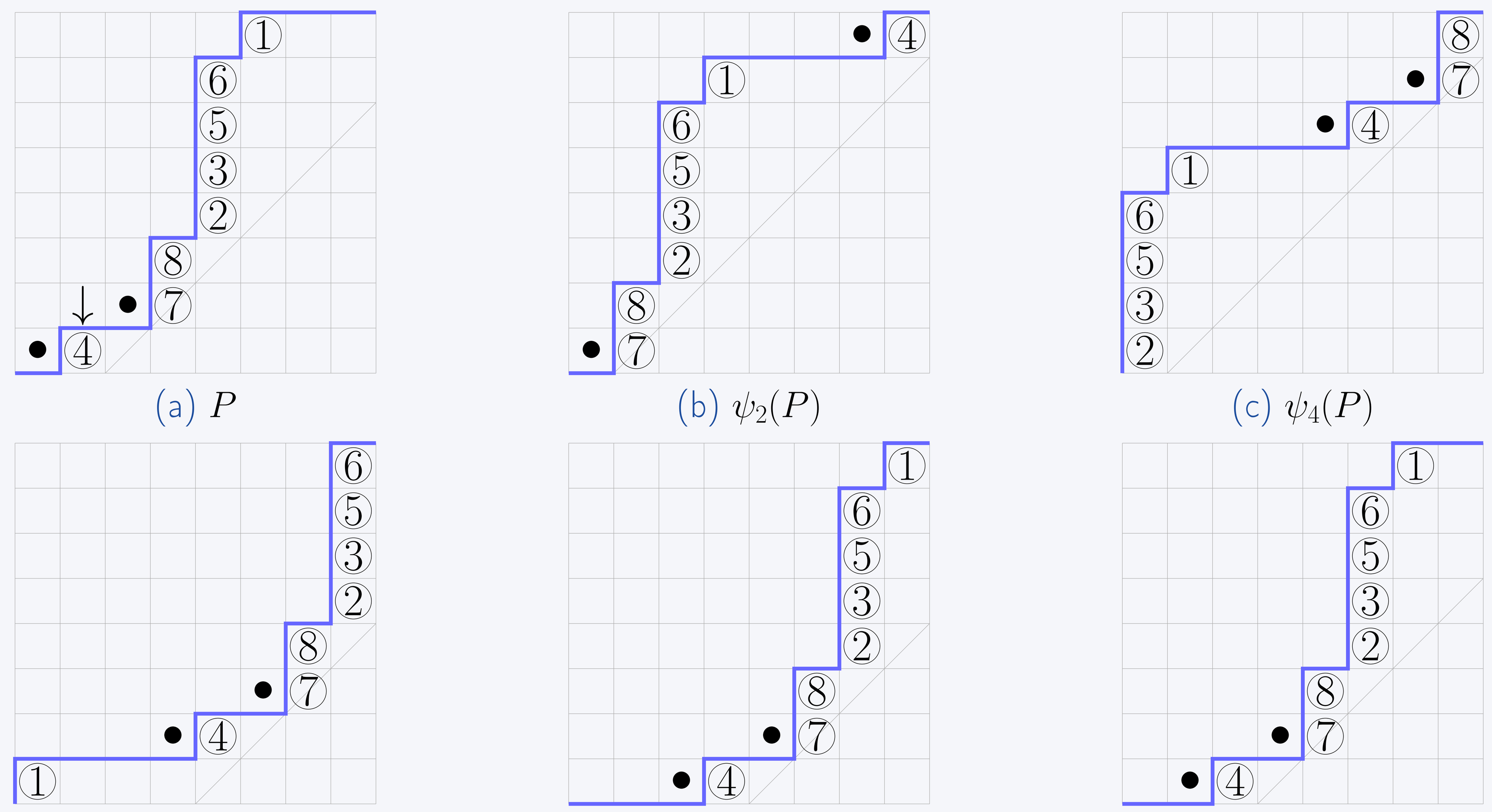
$\mathsf{stLSQ(n)}^{\bullet k}$ admits an action of C_{n-k} in terms of "cutting and pasting" operations. $S_{n,k}$ has cancellation-free expression in terms of equivalence classes under this action.

Split P into $P' \cdot P''$ after ith east step, $\psi_i(P) := P'' \cdot P'$.



(a) $P = \psi_3(P)$ (b) $\psi_1(P)$ (c) $\psi_2(P)$

Figure 2. The cyclic action on a decorated square path P. Its equivalence class is $\{P, \psi_1(P)\}$.



(a) P (b) $\psi_2(P)$ (c) $\psi_4(P)$ (d) $\psi_5(P)$ (e) $\psi_6(P)$ (f) $\psi_7(P)$

Figure 3. The equivalence class of a path P of schedule word 1^n

Theorem:

If $\mathsf{sched}(P) = 1^n$, there is an ordering $Q_0, \ldots, Q_{n-k-1}$ of this equivalence class such that $\mathsf{dinv}(Q_i) = i$.

[1] S. Corteel, M. Josuat-Vergès, and A. Vanden Wyngaerd, *Combinatorics of the delta conjecture at q=-1*, Algebraic Combinatorics **7** (2024), no. 1, 17–35.

[2] M. D'Adderio, A. Iraci, and A. Vanden Wyngaerd, *The Delta Square Conjecture*, International Mathematics Research Notices **2021** (2020), no. 1, 38–84.

[3] J. Haglund, J. Remmel, and A. Wilson, *The Delta Conjecture*, Transactions of the American Mathematical Society **370** (2018), no. 6, 4029–4057 (en).

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